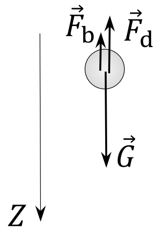
SOLUTIONS

Problem №1: Hydrodynamic drag

A) There are three forces acting on the ball:

1) The gravity in +Z direction: $G = mg$;
2) The buoyancy force in –Z direction: $F_\mathrm{b} = \rho_\mathrm{w}(4/3\,\pi R^3)g$;
3) The drag force in –Z direction, $F_\mathrm{d} = 1/2\,C_\mathrm{d}\rho_\mathrm{w}(\pi R^2)v^2$.



The Newton's II law for the Z-components states:

(1) $$ma = G - F_\mathrm{b} - F_d$$

which gives the following expression for the acceleration:

(2) $$a = g\left(1 - \frac{4\pi R^3\rho_\mathrm{w}}{3m}\right) - \frac{C_\mathrm{d}(\pi R^2)\rho_\mathrm{w}v^2}{2m}$$

B) We will consider v_i as initial velocity and will apply the equation of uniformly accelerated motion backward in time for the point z_{i-1} and forward for the point z_{i+1}:

(3) $$z_{i-1} = z_i - v_i\Delta t + 1/2\,a_i\Delta t^2$$

(4) $$z_{i+1} = z_i + v_i\Delta t + 1/2\,a_i\Delta t^2$$

Resolving the two equations with respect to v_i and a_i we get the final approximate expressions:

(5) $$v_i = \frac{z_{i+1} - z_{i-1}}{2\Delta t}$$

(6) $$a_i = \frac{z_{i-1} - 2z_i + z_{i+1}}{\Delta t^2}$$

The time interval between consecutive frames in our experiment is:

(7) $$\Delta t = 1/f = 0{,}20\ \mathrm{s}$$

By using equations (5) and (6) we can estimate the velocity and acceleration for frames #2 to #11, as given in the table below:

Frame #	z_i **(m)**	v_i **(m/s)**	a_i **(m/s²)**	v_i^2 **(m²/s²)**
1	0			
2	0,16	1,08	2,75	1,16
3	0,43	1,53	1,75	2,33
4	0,77	1,75	0,50	3,06
5	1,13	1,83	0,25	3,33
6	1,5	1,83	-0,25	3,33
7	1,86	1,83	0,25	3,33
8	2,23	1,83	-0,25	3,33
9	2,59	1,83	0,25	3,33
10	2,96	1,85	0,00	3,42
11	3,33	1,83	-0,25	3,33
12	3,69			

Velocity and acceleration for the first and the last frame cannot be calculated on the basis of our approximation since there are no coordinate measurements in earlier/later moments of time, as suggested in equations (5) and (6).

C) It follows from Eq. (2) that if we set a new independent variable $x = v^2$ and use $y = a$ as a dependent variable:

(8) $$y = px + q$$

with:

(9) $$p = -\frac{C_\mathrm{d}(\pi R^2)\rho_\mathrm{w}}{2m}$$

and

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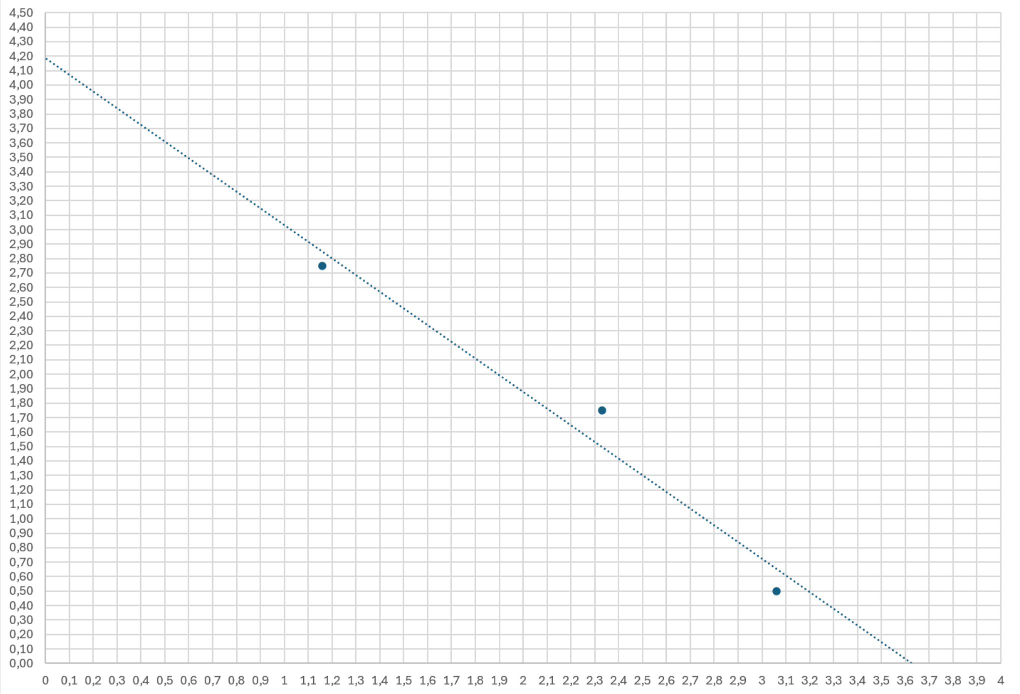
(10) $$q = g\left(1 - \frac{4\pi R^3 \rho_w}{3m}\right)$$

Since $y = a$ is already present in the table, we add only one new column in the table with values of the variable $x = v^2$. The corresponding *y*-*x* plot and the fitting line are given below:



Parameter q corresponds to the crossing point of the fitting line with the vertical axis:

(11) $$q \approx 4{,}2\ \mathrm{m/s^2}$$

and physically corresponds to the acceleration of the ball at zero velocity, i.e. just after it has been released from the clips. The crossing point with the horizontal axis corresponds to the square of the terminal velocity:

(12) $$v_t^2 \approx 3{,}6\ \mathrm{m^2/s^2}\ ;\ v_t \approx 1{,}9\ \mathrm{m/s}$$

Therefore the slope of the graph is:

(13) $$p = -q/v_t^2 \approx -1{,}2\ \mathrm{m^{-1}}$$

D) Since:

(14) $$\rho_{\mathrm{M}} = \frac{m}{4/3\,\pi R^3}$$

It follows from Eq. 10 that:

(15) $$q = g\left(1 - \frac{\rho_{\mathrm{w}}}{\rho_{\mathrm{M}}}\right)$$

and

(16) $$\rho_{\mathrm{M}} = \frac{\rho_{\mathrm{W}} g}{g - q} \approx 1{,}8\ 10^3\ \ \mathrm{kg/m^3}$$

After having calculated the density of the metal (most probably aluminum), we can estimate the radius of the ball through equation (14):

(17) $$R = \sqrt[3]{\frac{3m}{4\pi\rho_{\mathrm{M}}}} \approx 0{,}04\ \mathrm{m}$$

Now, the drag coefficient can be calculated from equation (9) for the slope p:

(18) $$C_{\mathrm{d}} = \frac{2m|p|}{\pi R^2 \rho_{\mathrm{w}}} \approx 0{,}29$$

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Marking scheme

Task	Element of the solution	Max. points
A	Diagram with all forces in correct directions	0,5
	Correct expressions for magnitudes of G, F_b and F_d	0,5
	Correctly written Newton's second law	0,5
	Final expression for acceleration	0,5
B	Equation (3) for uniformly accelerated motion backward in time	0,5
	Equation (4) for uniformly accelerated motion forward in time	0,5
	Gets equation (5) for velocity	1
	Gets equation (6) for acceleration	1
	Calculates time interval dt between two frames as 1/f	1
	Calculates to a precision 1 digits velocity for frames #2 to #11 (10 frames x 0,2 pt)	2
	Calculates to a precision 1 digits acceleration for frames #2 to #11 (10 frames x 0,2 pt)	2
	Anotates the "v" and "a" columns with titles and units	0,5
C	Proposes $x = v^2$ as auxiliary variable	1
	Expresses the slope p in terms of given parameters	1
	Expresses the constant q in terms of given parameters	1
D	Fills up a table column with calculated values of v^2 (10 frames x 0,1 pt)	1
	Anotates the v^2 column with title and unit	0,5
	Plots the (v^2,a) data points on a graph Points #2 to #4 since the terminal velocity (a=0) is reached	0,9
	Anotates plot axes with titles and units	0,6
	Draws axes tick marks with numeric labels	0,5
	Draws a fitting line	1
	Identifies constant q as a crossing point with the vertical axis and gets $q = 4,2\ m/s^2$	1
	Determines the slope $p = -1,2\ m^{-1}$	1
	Gets expression for ρ_M in terms of q	1
	Calculates ρ_M to a precision of $1,8.10^3\ kg/m^3$	1
	Expresses the ball radius by means of equation (17)	1
	Expresses the drag coefficient - equation (18)	1
	Calculates numerical value Cd = 0,29	1
	Total	**25**

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Problem №2: Trajectory

A) The motion of the first ball along the x-axis is uniform: $= v_0 \cos\alpha\, t$. (1) **(0.5 p)** The motion of the first ball along the y-axis is with constant acceleration a: $y = v_0 \sin\alpha\, t - \frac{1}{2}at^2$. (2) **(0.5 p)** Expressing t from Eq. (1) and substituting the result in (2): $= \tan\alpha\, x - \frac{ax^2}{2{v_0}^2\cos^2\alpha}$. **(1.0 p)** (3) Lets $A = \tan\alpha$ and $= \frac{a}{2{v_0}^2\cos^2\alpha}$. Then (3) is transformed to $y = Ax - Bx^2$. (4) Substituting in (4) the two positions of the first ball on the photos: $y_1 = Ax_1 - B{x_1}^2$ (5) and $y_2 = Ax_2 - B{x_2}^2$ (6). Multiplying Eq. (5) by ${x_2}^2$ and Eq. (6) by ${x_1}^2$ and subtracting them, we obtain $y_1{x_2}^2 - y_2{x_1}^2 = Ax_1x_2(x_2 - x_1)$. Then $A = \frac{y_1{x_2}^2 - y_2{x_1}^2}{x_1x_2(x_2-x_1)}$ (7). **(1.5 p)** Similarly, multiplying Eq. (5) by x_2 and Eq. (6) by x_1 and subtracting them, we obtain $y_1x_2 - y_2x_1 = Bx_1x_2(x_2 - x_1)$. Then $B = \frac{y_1x_2 - y_2x_1}{x_1x_2(x_2-x_1)}$. **(1.5 p)** (8) Finally, the formula for the trajectory is $y = \frac{y_1{x_2}^2 - y_2{x_1}^2}{x_1x_2(x_2-x_1)}x - \frac{y_1x_2 - y_2x_1}{x_1x_2(x_2-x_1)}x^2$. (9) **(1.0 p)**

B) $\tan\alpha = A = \frac{y_1{x_2}^2 - y_2{x_1}^2}{x_1x_2(x_2-x_1)} = \frac{9.15^2 - 21.5^2}{5.15(15-5)} = 2$. $\alpha \approx 63.4°$. **(1.0 p)** As $\frac{1}{\cos^2\alpha} \equiv \tan^2\alpha + 1$, then $B = \frac{a}{2{v_0}^2}(A^2+1)$ and $v_0 = \sqrt{\frac{a(A^2+1)}{2B}}$. (10) $B = \frac{y_1x_2 - y_2x_1}{x_1x_2(x_2-x_1)} = \frac{9.15 - 21.5}{5.15(15-5)} = 0.0040$ m^{-1}. Then using (10), $v_0 = \sqrt{\frac{10(2^2+1)}{2.0{,}04}} = 25{,}0$ m/s. **(2.0 p)**

C) From (3) it follows $0 = \tan\alpha\, L - \frac{aL^2}{2{v_0}^2\cos^2\alpha} = L\left(\tan\alpha - \frac{aL}{2{v_0}^2\cos^2\alpha}\right)$. **(0.5 p)** Then $L = \frac{2\tan\alpha\cos^2\alpha {v_0}^2}{a} = \frac{2\sin\alpha\cos\alpha {v_0}^2}{a}$. (11) **(0.5 p)** As $\tan\alpha = 2$, then $\cos\alpha = \frac{1}{\sqrt{5}}$ and $\sin\alpha = \frac{2}{\sqrt{5}}$. Substituting in (11), $L = \frac{2\frac{2}{\sqrt{5}}\frac{1}{\sqrt{5}}25^2}{10} = 50.0$ m. **(2.0 p)**

D) From Eq. (11) it follows $= \frac{2\sin\alpha\cos\alpha {v_0}^2}{a} = \frac{\sin(2\,\alpha){v_0}^2}{a}$. **(0.5 p)** As $L(\alpha) = L(\beta)$, then $\sin(2\,\alpha) = \sin(2\,\beta)$, $2\beta = \pi - 2\alpha$, $\beta = \frac{\pi}{2} - \alpha$. (12) **(1.5 p)**

E) From (1) $t_1 = \frac{x_1}{v_0\cos\alpha} = \frac{5}{25.\frac{1}{\sqrt{5}}} = \frac{1}{\sqrt{5}}$ s ≈ 0.45 s, **(1.5 p)** $t_2 = \frac{x_2}{v_0\cos\alpha} = \frac{15}{25.\frac{1}{\sqrt{5}}} = \frac{3}{\sqrt{5}}$ s $\approx 1{,}34$ s. **(1.5 p)**

F) Calculate the coordinates (x'_1, y'_1) and (x'_2, y'_2) of the second ball on the first and second photo, respectively.

From (1) and (12) $x'_1 = v_0\cos\beta\, t_1 = v_0\sin\alpha\, t_1 = \frac{v_0\sin\alpha\, x_1}{v_0\cos\alpha} = \tan\alpha\, x_1 = 2.5 = 10{,}0$ m. **(1.0 p)**

From (2) and (12) $y'_1 = \tan\beta\, x_1' - \frac{a{x_1'}^2}{2{v_0}^2\cos^2\beta} = \frac{x_1'}{\tan\alpha} - \frac{a{x_1'}^2}{2{v_0}^2\sin^2\alpha} = \frac{10}{2} - \frac{10.10^2}{2.25^2\left(\frac{2}{\sqrt{5}}\right)^2} = 4{,}0$ m. **(1.5 p)**

From (1) and (12) $x'_2 = v_0\cos\beta\, t_2 = v_0\sin\alpha\, t_2 = \frac{v_0\sin\alpha\, x_2}{v_0\cos\alpha} = \tan\alpha\, x_2 = 2.15 = 30{,}0$ m. **(1.0 p)**

From (2) and (12) $y'_2 = \tan\beta\, x_2' - \frac{a{x_2'}^2}{2{v_0}^2\cos^2\beta} = \frac{x_2'}{\tan\alpha} - \frac{a{x_2'}^2}{2{v_0}^2\sin^2\alpha} = \frac{30}{2} - \frac{10.30^2}{2.25^2\left(\frac{2}{\sqrt{5}}\right)^2} = 6{,}0$ m. **(1.5 p)**

g) (3 points) Calculate how long it took for the first ball to fall after the second (the time delay T).

From (1) $T = \frac{L}{v_0\cos\alpha} - \frac{L}{v_0\cos\beta}$ $\mathbf{(0.5\ p)}$ $= \frac{L}{v_0\cos\alpha} - \frac{L}{v_0\sin\alpha} = \frac{L}{v_0}\left(\frac{1}{\cos\alpha} - \frac{1}{\sin\alpha}\right)$ $\mathbf{(1.0\ p)}$ $= \frac{50}{25}\left(\sqrt{5} - \frac{\sqrt{5}}{2}\right) \approx 2{,}2$ s. **(1.5 p)**

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Problem №3: Istanbul Bosphorus Cross-Continental Swim and current correction

Choose the positive x-direction along the straight crossing from the Asian shore toward the European shore. Choose the positive y-direction upstream. Therefore the water current is

$$\vec{u} = (0, -u), \qquad u = 0.65\ \mathrm{ms^{-1}}.$$

The swimmer's speed relative to the water is constant:

$$v = 1.25\ \mathrm{ms^{-1}}.$$

The width of the crossing is

$$D = 1450\ \mathrm{m}.$$

(a) Angle required to land exactly at point B (6 points)

To land exactly at B, the swimmer must aim upstream so that the upstream component of his velocity relative to the water cancels the downstream current.

If the swimmer aims at angle θ above the line AB, the velocity components relative to the water are

$$v_x = v\cos\theta,$$

(1.5p) $$v_y = v\sin\theta.$$

The ground-velocity component in the y-direction is therefore

$$v_{y,\mathrm{ground}} = v\sin\theta - u.$$

For exact arrival at B,

(1.5p) $$v_{y,\mathrm{ground}} = 0.$$

Thus

$$v\sin\theta - u = 0$$

(1.5p) $$v\sin\theta = u,$$

(0.5p) $$\sin\theta = \frac{u}{v} = \frac{0.65}{1.25} = 0.52.$$

Therefore

(1.0p) $$\theta = \sin^{-1}(0.52) \approx 31.3°.$$

Hence the swimmer must aim approximately

$$\boxed{\theta \approx 31.3°\ \text{upstream.}}$$

(b) Time required to cross the Bosphorus (6 points)

The forward speed across the Bosphorus is the x-component of the ground velocity:

(2.0p) $$v_{x,\mathrm{ground}} = v\cos\theta.$$

Since

(1.0p) $$\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - (0.52)^2} = \sqrt{0.7296} \approx 0.8542,$$

we get

$$v_{x,\mathrm{ground}} = 1.25(0.8542) \approx 1.068\ \mathrm{m/s}.$$

The crossing time is

(2.0p) $$t = \frac{D}{v_{x,\mathrm{ground}}} = \frac{1450}{1.068} \approx 1.36 \cdot 10^3\ \mathrm{s}.$$

Thus

(1.0p) $$t \approx 1358\ \mathrm{s} \approx 22.6\ \mathrm{min}.$$

(c) Second swimmer who aims directly toward B

Now the swimmer does not correct for the current. Therefore the swimmer's velocity relative to the water is completely in the x-direction:

$$v_x = v = 1.25\ \mathrm{m/s}.$$

The current gives the whole y-component of the ground velocity:

$$v_y = -u = -0.65\ \mathrm{m/s}.$$

Time to reach the European shore. Since the x-component is $1.25\ \mathrm{ms^{-1}}$, the crossing time is

(2.0p) $$t_2 = \frac{D}{v} = \frac{1450}{1.25} = 1160\ \mathrm{s}.$$

Thus

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(1.0p) $\boxed{t_2 = 1160\ \text{s.}}$

Downstream displacement. The downstream drift is

(2.0p) $$d = ut_2 = 0.65(1160) = 754\text{m}.$$

Therefore the swimmer lands

(1.0p) $\boxed{754\text{m}}$

downstream from point B.

(d) Analysis of the GPS checkpoint data

For each swimmer, the average velocity components are obtained from

$$v_x' = \frac{x}{t}, v_y' = \frac{D}{t},$$

where x is the final displacement parallel to the current direction relative to point B.

Swimmer S1

$$t = 1320\text{s}, \qquad x = 0.$$

Hence

(1.0p) $$v_x' = \frac{0}{1320} = 0,$$

(1.0p) $$v_y' = \frac{1450}{1320} \approx 1.099\,\frac{\text{m}}{\text{s}}.$$

So

$$\boxed{v_x' = 0, \qquad v_y' \approx 1.099\ \text{ms}^{-1}.}$$

Swimmer S2

$$t = 1160\ \text{s},\ x = 270\ \text{m}.$$

Thus

(1.0p) $$v_x' = \frac{270}{1160} \approx 0.233\ \text{m/s},$$

(1.0p) $$v_y' = \frac{1450}{1160} \approx 1.25\,\frac{\text{m}}{\text{s}}.$$

Therefore

$$\boxed{v_x \approx 0.233\text{ms}^{-1}, \qquad v_y = 1.25\text{ms}^{-1}.}$$

Swimmer S3

$$t = 1450\ \text{s}, \qquad x = -110\ \text{m}.$$

Thus

(1.0p) $$v_x' = \frac{-110}{1450} = -0.0759\ \text{m/s},$$

(1.0p) $$v_y' = \frac{1450}{1450} \approx 1.00\ \frac{\text{m}}{\text{s}}.$$

Therefore

$$\boxed{v_x \approx -0.0759\ \text{ms}^{-1}, \qquad v_y = 1.00\ \text{ms}^{-1}.}$$

The best current correction corresponds to the swimmer who lands closest to the target line through B. This means the swimmer with the smallest value of $|x|$.

The displacements are

$$|x_{S1}| = 0, \quad |x_{S2}| = 270\text{m}, \quad |x_{S3}| = 110\text{m}.$$

Therefore the best current correction is achieved by

(1.0p) $\boxed{\text{S1.}}$

SOLUTIONS

Problem №4: Spring circuit

A) Capacitance

$C = \varepsilon_0 A / d$

When the distance becomes d/2:

$C' = \varepsilon_0 A / (d/2) = 2C$ 1p

$Q' = C' \Delta V = 2C \Delta V$ 1p

B) Force between the plates

$U = Q^2 / (2C)$

For d:

$U_1 = Q^2 d / (2 \varepsilon_0 A)$ 1p

For d/2:

$U_2 = Q^2 (d/2) / (2 \varepsilon_0 A)$ 1p

$\Delta U = - Q^2 d / (4 \varepsilon_0 A)$ 2p

$W = F d/2$ and $W = -\Delta U$ 6p

$F = Q^2 / (2 \varepsilon_0 A)$ 2p

C) Force and springs

Displacement:

$d/2 = F/k + F/k = 2F/k \rightarrow F = kd / 4$

Electric force:

$F = Q^2 / (2 \varepsilon_0 A)$

$Q=C' \Delta V$

$C'=2C$

$F=C'^2 \Delta V^2 / (2 \varepsilon_0 A)$

$F=(2C)^2 \Delta V^2 / (2 \varepsilon_0 A)$

Since $\varepsilon_0 A = Cd$:

$F = 2C (\Delta V)^2 / d$ 5p

Equation:

$kd / 4 = 2C (\Delta V)^2 / d \rightarrow k = 8C (\Delta V)^2 / d^2$ 1p

D. Energy

Initially:

$U = 1/2\, C\Delta V^2$

Finally:

$U' = 1/2 \cdot 2C \cdot \Delta V^2 = C\Delta V^2$

$\Delta U = 1/2\, C\Delta V^2$ 1p

Spring energy:

$U\_spring = 2 \cdot (1/2)\, k\, (d/4)^2 = kd^2 / 16$ 2p

Equating:

$1/2\, C\Delta V^2 = kd^2 / 16 \rightarrow k = 8C\Delta V^2 / d^2$ 2p